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General Dynamical Equations for Free Particles and Their Galilean Invariance

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Abstract Dynamical equations describing evolution of state functions in space-time of a given metric are important components of physical theories of particles. A method based on a group of the metric is used to obtain an infinite set of general dynamical equations for a scalar and analytical function representing free and spinless particles. It is shown that this set of equations is the same for any group of the metric that consists of an invariant Abelian subgroup of translations in time and space. For Galilean space-time, such group is the extended Galilei group. Using this group, it is proved that the infinite set of equations has only one subset of Galilean invariant dynamical equations, and that the equations of this subset are Schrödinger-like equations.

Keywords Theories of free particles · Galilei and extended Galilei groups · Schrödinger-like equations

1 Introduction

In a physical theory that describes free particles in space-time of a given metric, the particles must be represented by state functions whose evolution in space-time is given by a dynamical equation that must have the same form in all isometric frames of reference. Coordinate transformations that leave the metric unchanged are called isometric transformations and they form a group of the metric. A sufficient condition that two observers with the same metric (whose frames of reference differ by an isometric transformation of coordinates) identify the same physical object is that the state functions describing this object transform like one of the irreducible representations (irreps) of the group of the metric. This idea was first used by Wigner [1] who found the unitary irreps of the Poincaré group and used them to classify the elementary particles of physics [2].

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To formulate a fundamental physical theory of free and spinless particles in space-time of a given metric, a dynamical equation describing evolution of a state function of these particles must have the same form in all isometric frames of reference; in this paper, we refer to such equations as invariant dynamical equations. Different methods have been used to obtain such equations. In a standard approach, the classical concepts of energy and momentum become operators [3]. In another approach, Casimir operators of a group of the metric are used [2, 4]. There is also a Lagrangian formalism that requires an a priori knowledge of a Lagrangian [5, 6].

In previous studies of free and spinless particles in Galilean space-time, it was established that Schrödinger's equation [7] plays the central role in Galilean relativity because of its Galilean invariance [8–11]. Moreover, the method of Bargmann and Wigner [4] was used by Levy-Leblond [9] to obtain a set of Galilean invariant dynamical equations for free particles with arbitrary spins. In our work [12], we developed a method that is based on the Galilei group of the metric [8, 10] and demonstrated how to formally obtain Galilean invariant Schrödinger-like equations.

In this paper, a similar method (see Sect. 2) is used to derive an infinite set of dynamical equations that describe evolution of a scalar and analytic state function in time and space. The obtained results show that the equations are the same for any group of the metric that consists of an invariant Abelian subgroup of translations in time and space. For Galilean space-time, a group that has the required structure is the extended Galilei group. Using this group, a formal proof is given that only Schrödinger-like equations are Galilean invariant dynamical equations.

The paper is organized as follows: our method to obtain dynamical equations is described in Sect. 2; general dynamical equations for a scalar and analytic state function are derived in Sect. 3; Galilean invariant dynamical equations are obtained in Sect. 3; and conclusions are given in Sect. 4.

2 Method to Derive Dynamical Equations

The most fundamental problem of any physical theory is to obtain a dynamical equation that describes how a state function evolves in time and space. In the approach presented here, we specify the state function and develop a method that allows us to derive a set of general dynamical equations for this function. To select a subset of dynamical equations that have the same form in all isometric frames of reference, we specify the metric and search for dynamical equations that are invariant with respect to all coordinate transformations that leave the metric unchanged.

Let $\psi(\mathbf{r}, t)$, with $\mathbf{r} = (x, y, z)$, be a scalar and analytic function representing the particles. In general, this state function may have different forms in different frames of reference, however, observers with the known metric (whose frames of reference differ by an isometric transformation of coordinates) must be able to determine the transformation rule between the frames.

Our method is based on a group G of the metric that must have the following structure

$$G = T(3+1) \otimes_s H, \tag{1}$$

where $T(3+1)$ is an invariant Abelian subgroup of translations in space and time, H is a subgroup of G and $\otimes_s$ represents a semi-direct product.

Let us consider a set of N functions that forms a basis of an N-dimensional representation given by a set of $N \times N$ matrices A for each irreducible representation (irrep), and for each element of the group

$$\hat{\alpha} f_l^{(i)} = \sum_m A_{ml}(\hat{\alpha}) f_m^{(i)}, \tag{2}$$

where $\hat{\alpha}$ is one of the elements of the group, i labels the irreps and l is one of the members of the set of N functions satisfying (2). In addition, the sum on m is over the N members of the set, and the matrices A are assumed to be unitary. We write this equation for space and time translations separately, and obtain

$$\hat{T}_{\mathbf{a}} \psi(\mathbf{r}, t) \equiv \psi(\mathbf{r} + \mathbf{a}, t) = e^{i\mathbf{k}\cdot\mathbf{a}} \psi(\mathbf{r}, t), \tag{3}$$

and

$$\hat{T}_{t_0} \psi(\mathbf{r}, t) \equiv \psi(\mathbf{r}, t + t_0) = e^{-i\omega t_0} \psi(\mathbf{r}, t), \tag{4}$$

where $\mathbf{a}$ and t_0 represent translations in space and time, respectively. This shows that the unitary irreps of the group $T(3+1)$ are labeled by the real scalar ω and the real vector $\mathbf{k}$; note that there are no other restrictions on these quantities. It must also be mentioned that these labels are preserved in the irreps of G because $T(3+1)$ is its invariant subgroup.

Since $\psi(\mathbf{r}, t)$ is an analytic function, we may write

$$\psi(\mathbf{r} + \mathbf{a}, t) = e^{i(-i\mathbf{a}\cdot\nabla)} \psi(\mathbf{r}, t), \tag{5}$$

and

$$\psi(\mathbf{r}, t + t_0) = e^{-i(i\frac{\partial}{\partial t})} \psi(\mathbf{r}, t). \tag{6}$$

Comparing (3) and (5), and (4) and (6), we have

$$-i\nabla \psi(\mathbf{r}, t) = \mathbf{k} \psi(\mathbf{r}, t), \tag{7}$$

and

$$i\frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \omega \psi(\mathbf{r}, t). \tag{8}$$

The obtained eigenvalue equations are the necessary conditions for $\psi(\mathbf{r}, t)$ to transform like an irrep of G and to be an eigenfunction of the generators of the invariant Abelian subgroup $T(3+1)$.

Thus use of the irreps of G, properties of the generators of G and analyticity of $\psi(\mathbf{r}, t)$ has led us to the eigenvalue equations, which we consider to be our basic equations for developing a physical theory of free and spinless particles in space-time with the group G of the metric. Note that the eigenvalues ω and $\mathbf{k}$ label the irreps of G.

The above method is related to the one developed by us previously [12]. In the previous method, we introduced a new function $\eta(\mathbf{r}, t)$ that accounts for the fact that the eigenfunctions of the energy and momentum operators are not the same. However, when $\eta(\mathbf{r}, t) = 1$, the operators in (7) and (8) have the same eigenfunctions, which is the case considered in this paper.

3 General Dynamical Equations

Using the basic eigenvalue equations (see (7) and (8)), we derive a set of dynamical equations given by

$$i^m \frac{\partial^m \psi}{\partial t^m} - i^{2n} \frac{\omega^m}{k^{2n}} \nabla^{2n} \psi = 0, \tag{9}$$

where $k^2 = \mathbf{k} \cdot \mathbf{k}$, and m and n are positive integers. Analytic of $\psi(\mathbf{r}, t)$ guarantees that the time and space derivatives of arbitrary order can be considered, which gives infinitely many dynamical equations that could formally be used to construct physical theories.

In general, dynamical equations resulting from (9) can be divided into two classes, namely, equations that are symmetric ($m = 2n$) and asymmetric ($m \neq 2n$) in time and space. For odd values of m, all resulting dynamical equations are asymmetric, and for each even value of m there is only one symmetric equation. Note also that both sets of symmetric and asymmetric equations are infinite.

In the case of $m = n = 1$, we obtain a Schrödinger-like equation, and for $m = 2$ and $n = 1$ a wave-like equation is derived. Dynamical equations with higher-order spatial and time derivatives can also be obtained from (9) by considering higher values of m and n.

The fact that the set of dynamical equations given by (9) is valid for any metric whose group has the same structure as G (see (1)) is an important result of this paper. However, once the metric is known a subset of dynamical equations that are invariant with respect to all transformations that leave the metric unchanged must be determined. As already stated in Sect. 1, only the invariant dynamical equations of this subset can be used to formulate fundamental physical theories of particles.

Since the aim of this paper is to formulate fundamental physical theories in Galilean space-time, we now determine a subset of Galilean invariant dynamical equations from the infinite set of equations given by (9).

4 Galilean Invariant Dynamical Equations

4.1 Galilei and Extended Galilei Groups

In Galilean space-time, the metric is given by $ds_1^2 = dx^2 + dy^2 + dz^2$ and $ds_2^2 = dt^2$, where x, y and z are spatial coordinates and t is time. The group of this metric is called the Galilei group and its structure is

$$G_+^\uparrow = [T(1) \otimes R(3)] \otimes_s [T(3) \otimes B(3)] \tag{10}$$

where $T(1)$, $R(3)$, $T(3)$ and $B(3)$ are subgroups of translation in time, rotations, translations in space, and boosts, respectively. In addition, $\otimes$ is a direct product and $\otimes_s$ represents a semi-direct product. Note that the subgroup $[T(3) \otimes B(3)]$ is an invariant subgroup of $G_+^\uparrow$. The irreps of the group $G_+^\uparrow$ are the so-called vector irreps and they have no physical interpretation [13]. There also exists an infinite number of projective (ray) irreps of $G_+^\uparrow$, however, they are not equivalent to any vector irrep of the group [14]. These projective irreps can be obtained by using the method of induced representations [10, 15] that are characterized by a constant M, which is introduced as a phase factor in defining the irreps [8].

The process of introducing M is equivalent to the central extension of the associated Lie algebra [16] and the resulting group is called the extended Galilei group G_e; it must be noted that G_e is not the group of the metric but instead it corresponds to the extended Galilei metric. The structure of this group is

$$G_e = [R(3) \otimes_s B(3)] \otimes_s [T(3+1) \otimes U(1)] \tag{11}$$

where $T(3+1)$ is a subgroup of translations in space and time and $U(1)$ is a one-parameter unitary group [10, 15]. The group G_e is the universal covering group of the Galilei group $G_+^\uparrow$ and its irreps provide the projective irreps of $G_+^\uparrow$.

As shown above, the groups $G_+^\uparrow$ and G_e have different structures, namely, the subgroups $T(1)$ and $T(3)$ do not form a common invariant subgroup of $G_+^\uparrow$, however, the subgroup $T(3+1) = T(3) \otimes T(1)$ is an invariant Abelian subgroup of G_e. Hence, our method requires that G_e is used to determine a subset of Galilean invariant dynamical equations from the infinite set of equations given by (9).

4.2 Galilean Invariance

The form of (9) indicates that all resulting equations are invariant to translations and rotations, which form subgroups of the group G_e (see (1)). Another subgroup of G_e is formed by boosts. Hence, what remains to be done is to test all equations given by (9) for Galilean boost invariance. In Galilean space-time, a boost is defined by the change of coordinate frame of reference

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t \quad \text{and} \quad t' = t, \tag{12}$$

which gives

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \quad \text{and} \quad \nabla' = \nabla. \tag{13}$$

From (7) and (8), it is clear that neither $\mathbf{k}$ nor ω are invariant to Galilean boosts. This means that the only way to find Galilean invariant equations among all possible dynamical equations given by (9) is to assume that ω^m/k^{2n} is a Galilean invariant scalar, or $\omega^m/k^{2n} =$ constant. The main results of our search for Galilean invariant dynamical equations are presented by the following proposition.

Proposition *Let $\psi(\mathbf{r}, t)$ be a scalar and analytic function that satisfies*

$$i^m \frac{\partial^m \psi}{\partial t^m} - i^{2n} C_{m,2n} \nabla^{2n} \psi = 0, \tag{14}$$

where $C_{m,2n} = \omega^m/k^{2n}$, with m and n being positive integers. If $C_{m,2n}$ is assumed to be constant, then only one equation is Galilean invariant with $C_{1,2} = constant$.

Proof Our basic transformation adopted in this proof is $\psi(\mathbf{r}, t) = \phi(\mathbf{r}, t)\psi'(\mathbf{r}, t)$, where $\phi(\mathbf{r}, t)$ is an analytic function to be determined, and $\psi'(\mathbf{r}, t)$ is the transformed state function. Using this transformation and (12) and (13), we derive the transformed dynamical equation for $\psi'(\mathbf{r}', t')$. For this transformed equation to be Galilean invariant, it is required that its form is the same as the form of the dynamical equation for the original state function $\psi(\mathbf{r}, t)$; note that $C_{m,2n} =$ constant.

We vary m and n in (14), and seek dynamical equations for which the arbitrary function $\phi(\mathbf{r}', t')$ can be determined. If ϕ is found for given values of m and n, then the resulting dynamical equation is Galilean invariant and the transformation law for the state functions

$\psi(\mathbf{r},t)$ and $\psi'(\mathbf{r}',t')$ is also obtained; this invariant equation represents a possible physical theory in Galilean relativity. We begin the proof with the special case of $m=1$ and $n=1$, which represents Schrödinger-like equations. Then, we consider the case of $m=2$ and $n=1$, which corresponds to wave-like equations, and show that the results obtained for this case can be generalized to all higher values of m and n.

For the case of $m=1$ and $n=1$, the following transformed equation is obtained

$$i\frac{\partial\psi'}{\partial t'} + C_{1,2}\nabla'^2\psi' = 0, \tag{15}$$

where $C_{1,2}$ = constant. Since the form of this equation is the same as that of the original dynamical equation for $\psi(\mathbf{r},t)$ (see (14)), the condition that is required to determine the function $\phi(\mathbf{r},t)$ is given by

$$\left(i\frac{\partial\phi}{\partial t'} - i\mathbf{v}\cdot\nabla'\phi + C_{1,2}\nabla'^2\phi\right)\psi' + \left(2C_{1,2}\nabla'\phi - i\mathbf{v}\phi\right)\cdot\left(\nabla'\psi'\right) = 0. \tag{16}$$

Because $\psi' \neq 0$ and $\nabla'\psi' \neq 0$ for all x and t, the following independent conditions resulting from (16) are obtained

$$i\frac{\partial\phi}{\partial t'} - i\mathbf{v}\cdot\nabla'\phi + C_{1,2}\nabla'^2\phi = 0, \tag{17}$$

and

$$2C_{1,2}\nabla'\phi - i\mathbf{v}\phi = 0. \tag{18}$$

From (17) and (18), we have $\phi = \phi_0\exp[i(\mathbf{v}\cdot\mathbf{r}' + v^2t'/2)/2C_{1,2}]$, where ϕ_0 is an integration constant. The existence of the function $\phi(\mathbf{r}',t')$ is an important result because it demonstrates that Schrödinger-like equations are Galilean invariant and that the explicit form of the transformation law for the state functions $\psi(\mathbf{r},t)$ and $\psi'(\mathbf{r}',t')$ can be derived. Taking $\phi_0 = 1$, the transformation law becomes

$$\psi(\mathbf{r},t) = \psi(\mathbf{r}'+\mathbf{v}t',t') = \psi'(\mathbf{r}',t')e^{\frac{i}{2C_{1,2}}(\mathbf{v}\cdot\mathbf{r}'+\frac{1}{2}v^2t')}, \tag{19}$$

where $C_{1,2} = \omega/k^2$.

The second case considered in this proof is $m=2$ and $n=1$. The transformed dynamical equation for this case can be written in the following form

$$\frac{\partial^2\psi'}{\partial t'^2} + C_{2,2}\nabla'^2\psi' = 0, \tag{20}$$

where $C_{2,2}$ = const, and the condition that must be satisfied by the function $\phi(\mathbf{r}',t')$ is

$$\begin{aligned}
&\left[\frac{\partial^2\phi}{\partial t'^2} - 2(\mathbf{v}\cdot\nabla')\frac{\partial\phi}{\partial t'} + (v^2 - C_{2,2})\nabla'^2\phi\right]\psi' \\
&+ 2\left[\frac{\partial\phi}{\partial t'} - (\mathbf{v}\cdot\nabla')\phi - \phi(\mathbf{v}\cdot\nabla')\right]\left(\frac{\partial\psi'}{\partial t'}\right) \\
&+ 2\left[(v^2 - C_{2,2})(\nabla'\psi')\left(\frac{\partial\phi}{\partial t'}\right)\mathbf{v}\right]\cdot(\nabla'\psi') \\
&+ \left[v^2\phi\right](\nabla'^2\psi') = 0.
\end{aligned} \tag{21}$$

Assuming that $\psi' \neq 0$, $\partial\psi'/\partial t \neq 0$, $\nabla'\psi' \neq 0$ and $\nabla'^2\psi' \neq 0$ for all x and t, we have the four conditions on the function ϕ. One of these conditions requires that ϕv^2 must be equal to zero. Since $v^2 \neq 0$, we must have $\phi = 0$, which means that no transformation law between the state functions $\psi(\mathbf{r}, t)$ and $\psi'(\mathbf{r}', t')$ exists. As a result, the wave-like equation given by (20) is not Galilean invariant.

The above result can be generalized to all cases of $m > 2$ and $n > 1$, and used to demonstrate that none of the dynamical equations obtained from (14) with the above values of m and n is Galilean invariant. To prove the latter, we recall that $\nabla^{2n}\psi$, with $n > 1$, produces all possible combinations of the function ϕ and its derivatives with the function ψ' and its derivatives. One of these combinations will always give the following condition: $[c_0\phi](\nabla'^{2n}\psi') = 0$, where c_0 is a constant. Since $c_0 \neq 0$ and $\nabla'^{2n}\psi' \neq 0$ for all x, the function ϕ must be zero. The result has profound implications, namely, it shows that none of the dynamical equations obtained with $m > 2$ and $n > 1$ is Galilean invariant. This concludes our proof. □

4.3 Discussion

The main result of the above proposition is that theories of free elementary particles that are described by scalar, analytical functions in Galilean space-time can only be formulated by functions which satisfy Schrödinger-like equations with $m = n = 1$ and $C_{1,2} = \omega/k^2 =$ constant (see (14)). This is an important result as it shows that no fundamental physical theory based on dynamical equations with derivatives higher than the first-order derivative in time and higher than the quadratic derivative in space can be formulated in Galilean space-time. It is interesting to note that this restriction includes wave-like equations, at least, for scalar state functions; on the other hand, it is already known that Galilean invariant dynamical equations for non-scalar functions exist [8–10].

The fact that the results of our proposition are only valid for $C_{1,2} = \omega/k^2 =$ constant has significant consequences because the coefficient $C_{1,2}$ must be identified with a physical quantity that remains the same in both stationary and moving frames of reference. For free particles considered here, such quantity can be their mass; actually, we denote this mass as M and refer to it as 'scaled mass' because of its physical units. Hence, we may write $C_{1,2} = 1/2M$ or $M = k^2/2\omega$, which is an interesting relationship between the scaled mass M and the eigenvalues ω and k.

The physical interpretation of the constant M was also given by Levy-Leblond [8], however, his approach was different. He started with Schrödinger's equation [7] of non-relativistic quantum mechanics and demonstrated that under Galilean transformations the eigenfunctions must transform like the projective irreps of $G_+^\uparrow$. After performing Fourier transforms, he showed that M corresponds to particle's mass in Schrödinger's equation [9]. He did not derive Schrödinger's equation but instead he showed that it was a Galilean invariant equation [8, 10, 11].

Let us substitute $C_{1,2} = 1/2M$ into (14) and (15), and obtain Schrödinger-like equations for $\psi(\mathbf{r}, t)$ and $\psi'(\mathbf{r}', t')$, respectively; since both equations are Galilean invariant, their forms are the same in the stationary $(\mathbf{r}, t)$ and moving $(\mathbf{r}', t')$ frames of reference. By substituting $C_{1,2} = 1/2M$ into (19), one obtains the explicit form of the transformation law for the state functions $\psi(\mathbf{r}, t)$ and $\psi'(\mathbf{r}', t')$. The observers can use this law to determine how the state function in one coordinate system is related to that in the other system [3].

Among Schrödinger-like equations, the most prominent member of this class is Schrödinger's equation of non-relativistic quantum mechanics. For this equation, we have $C_{1,2} = \hbar/2m_0$ or $M = m_0/\hbar$, where m_0 is mass of an elementary particle. We may also write

$m_0 = \hbar k^2/2\omega$, which shows the dependence of particle's mass on the Planck constant $\hbar$ and the eigenvalues k and ω.

A standard approach to introduce Schrödinger's equation for free particles with zero spin is to assume that the classical concepts of energy and momentum become operators in the respective energy and momentum relationships [3]. There is also an elegant procedure of obtaining Schrödinger's equation by using a Lagrangian formalism, which requires an a priori knowledge of Lagrangian [5, 6]. Unfortunately, currently no first principle theory exists that would allowed predicting the form of the required Lagrangian [5].

Another approach is to use Casimir operators of the symmetry group of Schrödinger's equation, which happens to be the extended Galilei group G_e. The Casimir operators of this group are known [6, 10] and the eigenvalue equation for one of these operators gives directly Schrödinger's equation. The problem with this approach is that it requires an a priori knowledge of the structure of G_e. Actually, the group G_e and its Lie algebra were established using the already known form of Schrödinger's equation [8, 10, 15].

Clearly, the above methods to obtain Schrödinger's equation are different than the method introduced in Sect. 2. The main advantage of our method is that it allows deriving dynamical equations regardless whether Casimir operators and Lagrangians are known or not. The method is based on the requirement that the group of a metric has the same structure as the group G given by (1), that a state function representing free particles is analytic, and that a dynamical equation describing evolution of this function in time and space is Galilean invariant. The two latter requirements can be respectively called the Principle of Analyticity and the Principle of Galilean Relativity [12]. The results of this paper show that both principles are required to obtain Schrödinger's equation.

5 Conclusions

A method based on a group of the metric was used to derive an infinite set of general dynamical equations for a scalar function representing free and spinless particles. The obtained results showed that this set of equations is the same for any group of the metric that has the same structure as the group $G = T(3+1) \otimes_s H$. The fundamental property of the group G is that $T(3+1)$ is an invariant Abelian subgroup of translations in space and time; note that there is no restriction on H, which is a subgroup of all other transformations that leave the metric invariant.

For Galilean space-time, a group that has the required structure is the extended Galilei group. Using this group, we proved that among infinitely many possible dynamical equations only Schrödinger-like equations are Galilean invariant. Hence, only Schrödinger-like equations can be used to formulate fundamental physical theories of particles in Galilean relativity.

The requirement that the state function is analytic, and that the dynamical equation describing evolution of this function in time and space is Galilean invariant can be called the Principle of Analyticity and the Principle of Galilean Relativity [12], respectively. An important result of this paper is that both principles are required to derive Schrödinger's equation.

Another example of a group that has similar structure as the group G is the Poincaré group, which is the group of the Minkowski metric [2, 8]. This means that our method can directly be used to formulate physical theories of free and spinless particles in Minkowski space-time. Actually, the infinite set of equations given by (9) is also valid for the Minkowski metric, however, a subset of invariant dynamical equations must still be selected.

Other metrics can also be considered, however, specific applications of our method to either the Minkowski metric and or other metrics is out of scope of this paper. It is important to mention that different metrics may lead to different fundamental physical theories and different sets of free particles that are allowed to be called elementary in these theories. In other words, what is a fundamental theory and elementary particle for one metric may not be true for another metric.

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